

Shear-wave speeds and elastic moduli for different liquids. Part 1. Theory

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In this paper we develop a theory for a rheometrical device for measuring the speed of shear waves into a region at rest. The device is a Couette apparatus with a narrow gap. The outer cylinder is moved impulsively and a time of transit is measured. The linearized theory governing this apparatus is reduced to a perturbation of Stokes' first problem between parallel planes. A method for determining an effective shear modulus from measured values of the wave speed is discussed and various cases are analysed. An experimental apparatus based on this theory, together with tabular data, is discussed in a companion paper (Part 2, Joseph, Riccius & Arney 1986).

1. Constitutive equations

There is a great simplification in the problem of constitutive modelling when the deformations are a small perturbation of states of rest. These deformations depend on a Newtonian viscosity μ and a smooth relaxation function $G(s)$, where $G(s) > 0$, $G'(s) < 0$ for $0 \leq s = t - \tau < \infty$ and τ is the past time. The stress τ is given by

$$\tau = 2\mu \mathbf{D}[\mathbf{u}] + 2 \int_0^\infty G(s) \mathbf{D}[\mathbf{u}(\mathbf{x}, t-s)] ds, \quad (1.1)$$

where $\mathbf{u}(\mathbf{x}, t)$ is the velocity and $\mathbf{D}[\mathbf{u}]$ is the symmetric part of the velocity gradient. Equation (1.1) is a Jeffreys' type of generalization of Boltzmann's equation of linear viscoelasticity in which the presence of a Newtonian viscosity is acknowledged. Equation (1.1) also holds in the class of small perturbations of rigid motions.

A constitutive equation of the rate type may be obtained as the time derivative of (1.1):

$$\frac{\partial \tau}{\partial t} = 2\mu \frac{\partial \mathbf{D}}{\partial t} + 2G(0) \mathbf{D} + 2 \int_0^\infty G'(s) \mathbf{D}[\mathbf{u}(\mathbf{x}, t-s)] ds. \quad (1.2)$$

Jeffreys' model is a special case of (1.2) in which

$$G(s) = \frac{\eta}{\lambda} e^{-s/\lambda}, \quad (1.3)$$

where λ is the relaxation time and η is the elastic viscosity. Combining (1.2) and (1.3) we get

$$\lambda \frac{\partial \tau}{\partial t} = 2\mu\lambda \frac{\partial \mathbf{D}}{\partial t} + 2\tilde{\mu} \mathbf{D} - \tau, \quad (1.4)$$

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where $\tilde{\mu} = \mu + \eta$. A retardation time

$$A = \frac{\mu\lambda}{\tilde{\mu}} \quad (1.5)$$

is usually defined for (1.4). When $\mu = 0$, (1.4) gives rise to a Maxwell model

$$\lambda \frac{\partial \boldsymbol{\tau}}{\partial t} = 2\eta \mathbf{D} - \boldsymbol{\tau}. \quad (1.6)$$

Fluids with $\mu = 0$ are like relaxing elastic solids: they propagate shock waves. Fluids with $\mu \neq 0$ are diffusive: they smooth shocks (cf. §3).

Equations (1.1) and (1.2) are perturbation equations and naturally not invariant under changes of frame that do not satisfy the same conditions of linearization. Various invariant theories that are said to be linear have been proposed. For example, Coleman & Noll (1961) linearized a functional depending on the history of the right relative Cauchy–Green strain tensor. Naturally they arrive at a linear expression, linear in this nonlinear tensor. They call this ‘the finite linear theory of viscoelasticity’. When applied to the incompressible fluids they get (1.1) with $\mu = 0$ and \mathbf{D} replaced by $\mathbf{C}_t(\mathbf{x}, t-s) - \mathbf{1} = \mathbf{G}(s)$, $\mathbf{G}(0) = \mathbf{0}$. The linearization of $\mathbf{G}(s)$ around $\mathbf{0}$ is \mathbf{D} (see Joseph 1976, Chap. XIII). If the relaxation function $G(s) = (\eta/\lambda) e^{-s/\lambda}$ is of Maxwell’s type, then Coleman & Noll’s is a lower convected Maxwell model. If we suppose that the stress functional depends on the Finger tensor, rather than the Cauchy tensor, we arrive at Lodge’s theory, which is the same as an upper convected Maxwell model when the relaxation function is of Maxwell’s type. Saut & Joseph (1983), under different hypotheses than Coleman & Noll, arrived at (1.1) with $\mathbf{G}(s)$ in the place of \mathbf{D} under the integral and $\mu \neq 0$. If Saut & Joseph had used $\mathbf{H}(s) = \mathbf{C}_t^{-1}(\mathbf{x}, t-s) - \mathbf{1}$ instead of $\mathbf{G}(s)$ they would have $\mathbf{H}(s)$ replacing \mathbf{D} under the integral. The rate equation for an equation of the Saut–Joseph type in \mathbf{H} with an exponential relaxation function is an Oldroyd B model. None of these so-called linear equations are completely linearized. When they are completely linearized they reduce universally to (1.1) and (1.2). These two equations are model independent: they apply to all viscoelastic fluids in motions which perturb rest. This shows that the Newtonian viscosity μ and the relaxation function $G(s)$ are genuine material parameters which are also model independent.

To our knowledge, the first person to introduce a rate equation with a Newtonian viscosity and relaxing elasticity was Jeffreys (1929, p. 265). Most of the models arising from molecular modelling of polymeric solutions have a Newtonian contribution from the solvent and are of the Jeffreys’ type. An invariant formulation of rate equations containing relaxation and retardation (Newtonian viscosity) effects evidently first appears in the celebrated 1950 paper of Oldroyd. Green & Rivlin (1960) appear to have been the first to introduce Newtonian viscosity into integral models. They get rate terms from integrals by allowing delta functions and their derivatives in the kernels. Saut & Joseph (1983) derived integral expressions of the type introduced by Green & Rivlin from a theory of fading memory in which the ensemble of all possible linearized stresses coincides with a certain topological dual of a domain space (say, a Sobolev space) for allowed deformations. Maxwell models and the generalization of these embodied in the theory of fading memory of Coleman & Noll (see Saut & Joseph for references) cannot contain a Newtonian viscosity. These models are all instantaneously elastic. Various kinds of hyperbolic phenomena, waves, shock waves, loss of evolution, Hadamard instabilities, and change of type arise in fluids with instantaneous elasticity (see Joseph, Renardy & Saut 1985;

Joseph & Saut 1986 for recent reviews). Many distinguished scientists of the 19th and early 20th centuries, such as Poisson, Maxwell, Poynting and Boltzmann, believed that liquids were closer to solids than to gases, with instantaneous and relaxing elasticity, and there is also a line of interesting experiments of the same period which explore this idea (see Joseph 1986 for a recent historical perspective).

It is desirable to characterize the static or zero-shear-rate viscosity for (1.1) in the following way. Assume steady shearing with one component of velocity $u(x)$ depending on one variable x . The shear stress $\tau(\kappa) = \tau_{12}$ of τ depends then on the rate of shear $\kappa(x) = D_{12}$ of \mathbf{D} and (1.1) reduces to $\tau = (\mu + \eta)\kappa$, where

$$\tilde{\mu} = \mu + \eta \quad (1.7)$$

is the zero-shear viscosity and

$$\eta = \int_0^\infty G(s) ds \quad (1.8)$$

is the elastic viscosity. Newtonian fluids have $\eta = 0, \tilde{\mu} = \mu$. Elastic fluids have $\mu = 0, \tilde{\mu} = \eta$. In general

$$\tilde{\mu} \geq \eta, \quad (1.9)$$

with equality for elastic fluids. It is easy to measure the zero-shear viscosity $\tilde{\mu}$, but the measurement leaves μ and η undetermined.

To decide about elasticity and viscosity we could consider ever more dilute solutions of polymer chains of large molecules in solvents which are thought to be Newtonian. What happens when we reduce the amount of polymer? There are two good ideas about this that are in collision. The first (Jeffreys') idea says that there is always a viscosity and some elasticity with an ever greater viscous contribution as the amount of polymer is reduced. The theories of Rouse and Zimm (see for example, Bird, Armstrong & Hassager 1977) adopt this view. On the other hand we may suppose that the liquid is elastic so that $\mu = 0$ and the static viscosity $\tilde{\mu} = \eta$ is the area under the graph of the relaxation function. Since η is finite in all liquids, we have $\eta = G(0)\bar{\lambda}$, where $\bar{\lambda}$ is a mean relaxation time. Maxwell's idea is that the limit of extreme dilution is such that the rigidity $G(0)$ tends to infinity and $\bar{\lambda}$ to zero in such a way that their product η is finite. Ultimately, when the polymer is gone, we are left with an elastic liquid with an enormously high rigidity. This idea apparently requires anomalous behaviour because $G(0)$ appears to decrease with polymer concentration when the concentration is finite.

The contradiction between the two foregoing ideas and the apparent anomaly can be resolved by replacing the notion of a single mean relaxation time with a distribution of relaxation times. This notion is well grounded in structural theories of liquids in which different times of relaxation correspond to different modes of molecular relaxation. It is convenient again to think of polymers in a solvent, but now we can imagine that the solvent is elastic, but with an enormously high rigidity. In fact many of the so-called Newtonian solvents have a rigidity $G(0)$ of the order 10^9 Pa, which is characteristic of glass, independent of variations of the chemical characteristics among the different liquids (for example, see Harrison 1976). To find this glassy modulus it is necessary to use high-frequency devices operating in the range of 10^9 Hz. These devices were first introduced by Mason (1947) and Mason *et al.* (1949) at the Bell laboratories. To measure the glassy modulus in low-molecular-weight liquids it is also necessary to super-cool them to temperatures near the glassy state. In these circumstances the liquid acts like a glassy solid, the molecular configurations cannot follow the rapid oscillations of stress, and the liquid cannot

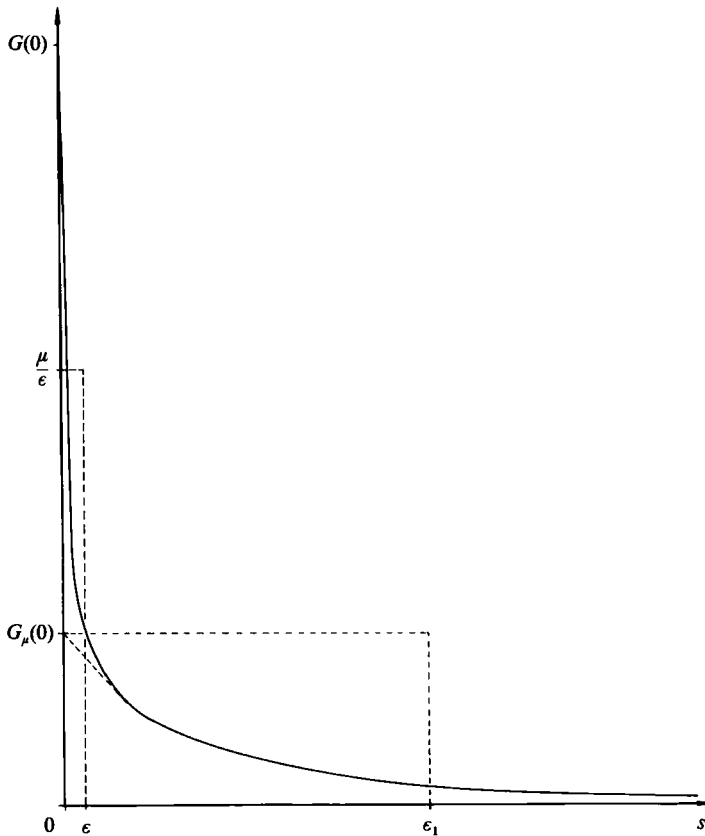


FIGURE 1. Typical relaxation function for a polymer solution. At very short times the molecular order is frozen and the liquid does not flow. The wave instantaneously sees a material that looks like an organic glass. Then the stress associated with the molecular forces between small molecules relaxes. The relaxation of larger structures, like those associated with macromolecules, is much slower with an effective shear modulus more typical of soft rubbers. Eventually all the elastic response decays in a diffusive limit with a viscosity given by the area under the graph of $G(s)$. The effective modulus $G_\mu(0)$ can be identified with the measured (Part 2) shear modulus $G_\epsilon = G(\epsilon)$. We define an effective relaxation function $G_\mu(s)$ with the tangent construction. First put $G_\mu(0) = G_\epsilon$; then drop the tangent to $G(s)$. This relaxation function may be modelled by a double step. The decay of the glassy modes under the first step gives rise to a 'Newtonian' viscosity μ . The viscosity associated with the second step is $G_\mu(\epsilon_1 - \epsilon)$, much larger than μ . The total viscosity is $\tilde{\mu} = \mu + G_\mu(\epsilon_1 - \epsilon)$. The usual type of estimate for the time of relaxation $\bar{\lambda}G(0) = \tilde{\mu}$ would lead one to a grossly misleading value $\bar{\lambda}$ of the true time of relaxation.

flow. For slower processes it is possible for the liquid to flow and if the relaxation is sufficiently fast the liquid will appear to be Newtonian in more normal flows. For practical purposes there is no difference between Newtonian liquids and liquids with rigidities of order 10^9 Pa and mean relaxation times of about 10^{-10} s. In fact it is convenient to regard such liquids as Newtonian, even though $\mu = 0$ and $\tilde{\mu} = \eta$.

The presence of polymers would not allow the liquid to enter the region of viscous relaxation at such early times. Instead much slower relaxation processes associated with the polymers would be induced (cf. Mason *et al.* 1949). The second slower regime of relaxation starts in a neighbourhood of small times which terminate fast relaxation from the glassy state. A value $G_\mu(0)$ representative of $G(t)$ at these early times can be said to loosely define a second smaller modulus, an effective modulus (see figure

1). For many polymer solutions this modulus defines the effective rigidity of the solution against changes of configuration of chains of polymers. It can be said that the effective modulus is associated with times so long that fast modes have relaxed and so short that slower modes have not.

The theory that will be developed in the rest of this paper is a linear one based on the linearized constitutive equation (1.1). There is some indication from the experiments described in Part 2 (Joseph *et al.* 1986) that nonlinear effects are small under the operating conditions of the wave-speed meter.

A list of symbols with meanings that are not obvious will be found in §8.

2. The relaxation function and viscosity

This section is a preparation for the theory and applications of the wave-speed meter. Since the meter is used to characterize some rheological properties of a liquid it is necessary to eschew special representations of the relaxation function. The theory in this paper is based on the linearized constitutive equation (1.1). Following ideas already outlined at the end of §1, we shall call μ the effective (Newtonian) viscosity and $G_\mu(s)$, the effective rigidity. To place this effort in context we need to know more about conventional methods for measuring μ and $G(s)$.

2.1. Viscosity

Although it is very easy to measure the zero-shear-rate viscosity $\tilde{\mu} = \mu + \eta$, direct measurement of the Newtonian viscosity μ seems to be a problem of great difficulty. If it were possible to compute the elastic viscosity η as the area under the relaxation curve we could find $\mu = \tilde{\mu} - \eta$. In fact methods for measuring the relaxation function are inadequate for small and large values of s , so that the computation of the area under the curve is problematic.

In fact, it may be true that $\mu = 0$, $\tilde{\mu} = \eta$ in every liquid, without exception. Then the effective viscosity would be associated with rapidly decreasing glassy modes excited only in rapid deformations. There is an impressive experimental literature using state-of-the-art high-frequency devices that supports this view. We lean to the view that $\mu = 0$ in exact sense, but $\mu \neq 0$ on the timescales relevant for applications. All this needs to be explained. For the moment let it be said that in view of the extraordinary effectiveness of the theory of Newtonian viscosity it would be foolish to put $\mu = 0$ in (1.1), even if, in fact, $\mu = 0$.

2.2. Stress relaxation

Simplified theories for measuring $G(s)$ using impulsive changes of deformation are described in the book by Bird *et al.* (1977). Inertia is neglected in the simplified theory but it is not negligible near $t = 0$. A theory that accounts for inertia is given in the paper by Narain & Joseph (1983*a*). The neglect of inertia in the simplified theory used for step strain experiments on commercial cone and plate rheometers may be justified because these rheometers have a response time of about 0.01 s and do not give data for small times where inertia is most important. The working formula for these rheometers is (see, for example, Bird *et al.* 1977 p. 284)

$$M = \frac{2}{3}\pi R^3\gamma_0 G(t), \quad (2.1)$$

where M is the torque, R is the radius of the sample in the apparatus, γ_0 is the shear strain and $G(t)$ the relaxation function.

2.3. *Small-amplitude oscillations*

Small-amplitude, sinusoidal oscillation devices are designed to measure functionals on propagating plane waves. Inertia is not neglected. Since the governing equations are linear, it is possible to superpose plane waves

$$u(y, t) = U e^{i(\omega t - \beta y)}, \tag{2.2}$$

where ω is the frequency of the imposed sinusoidal oscillation and $\beta = \gamma + i\alpha$ is to be determined as a solvability condition for the reduced equations of motion governing the amplitude. (The condition is that $\beta^2/\omega^2 = \rho/\tilde{G}^*(\omega)$.) The viscosity μ and relaxation function $G(s)$ enter these equations in a unique composition called a complex viscosity (see, for example, Bird *et al.* 1977)

$$\eta^*(\omega) = \mu + \int_0^\infty G(s) e^{-i\omega s} ds \stackrel{\text{def}}{=} \eta' - i\eta'' \tag{2.3}$$

The equations are satisfied when the phase speed $V(\omega) = \gamma/\omega$ is given by $V(\omega) = [2\omega q^2/\rho(\eta'' + q)]^{1/2}$ and the attenuation by $\alpha(\omega) = -[\omega\rho\eta'^2/2q^2(\eta'' + q)]^{1/2}$, where $q^2 = \eta'^2 + \eta''^2$. It is also useful to define the complex rigidity $\tilde{G}(\omega)^* \stackrel{\text{def}}{=} i\omega\eta^* = \tilde{G}' + i\tilde{G}''$, where $\tilde{G}'' = \omega\eta'(\omega)$ is the loss modulus, η' is the dynamic viscosity, and $\tilde{G}' = \omega\eta''(\omega)$ is the storage modulus. The storage modulus is an imperfect measure of elasticity (cf. discussion of §4, Part 2) and vanishes identically for Newtonian liquids. Asymptotic forms of $\eta^*(\omega)$ can be obtained by repeated integration by parts. Thus

$$\eta' = \mu - \frac{G'(0)}{\omega^2} + O\left(\frac{1}{\omega^4}\right), \quad \omega\eta'' = G(0) + O\left(\frac{1}{\omega^2}\right). \tag{2.4}$$

The limiting value of the storage modulus is the glassy modulus $G(0)$. When $\mu \neq 0$, the limit $\omega \rightarrow \infty$ leads to unbounded values of the phase speed and the attenuation (cf. (2.9)). On the other hand, when $\mu = 0$

$$\lim_{\omega \rightarrow \infty} [V(\omega), \alpha(\omega)] = \left[c, \frac{G'(0)}{2cG(0)} \right], \tag{2.5}$$

where $c = (G(0)/\rho)^{1/2}$ is the speed and $G'(0)/2cG(0)$ is the attenuation for shock waves of vorticity (shear) into liquids at rest.

2.4. *Spectral decomposition of $G(s)$; effective moduli*

We may understand glassy moduli, effective moduli, viscosity, fast and slow modes through a spectral decomposition of $G(s)$ in the instantaneously elastic case, $\mu = 0$. It suffices here to consider a finite basis of N modes of relaxation

$$\left. \begin{aligned} G(s) &= \sum_1^N G_k(s), \\ G_k(s) &= \frac{\eta_k}{\lambda_k} e^{-s/\lambda_k}, \quad \lambda_k > \lambda_{k-1}, \\ \tilde{\mu} = \eta &= \sum_1^N \eta_k. \end{aligned} \right\} \tag{2.6}$$

Suppose that ϵ is a small time defined by a physical process of interest and $\epsilon = \lambda_M$. Then, for short times $t \ll \lambda_M$ all the modes $k \geq M$ are glassy. For $t > \lambda_M$ all the modes

with $k < M$ have decayed. The effective (Newtonian) viscosity μ and rigidity $G_\mu(s)$ may then be defined as follows:

$$\left. \begin{aligned} \tilde{\mu} &= \mu + \eta, \\ \mu &= \sum_1^{M-1} \eta_k, \quad \eta = \sum_M^N \eta_k, \\ G_\mu(s) &= \sum_M^N G_k(s). \end{aligned} \right\} \quad (2.7)$$

For many polymeric solutions $\mu/\tilde{\mu}$ is very small and the contributions of the fast modes may be expressed by a relatively small Newtonian viscosity.

Let us now suppose that we have modelled the relaxation function so that the fast modes are expressed by a small viscosity and the slow ones by a single relaxation time. Then

$$\left. \begin{aligned} \eta^*(\omega) &= \mu + \frac{\eta}{1 + \lambda^2 \omega^2} - \frac{i\lambda\omega\eta}{1 + \lambda^2 \omega^2}, \\ V^2(\omega) &= \frac{2\omega}{\rho} \frac{\tilde{\mu}^2 + \mu^2 \lambda^2 \omega^2}{\lambda\omega\eta + (1 + \lambda^2 \omega^2)^{\frac{1}{2}} (\tilde{\mu}^2 + \mu^2 \lambda^2 \omega^2)^{\frac{1}{2}}}, \\ \alpha^2(\omega) &= \frac{\omega\rho[\tilde{\mu} + \mu\lambda^2\omega^2]^2}{2[\tilde{\mu}^2 + \mu^2\lambda^2\omega^2][\lambda\omega\eta + (1 + \lambda^2\omega^2)^{\frac{1}{2}}(\tilde{\mu}^2 + \mu^2\lambda^2\omega^2)^{\frac{1}{2}}]} \end{aligned} \right\} \quad (2.8)$$

Large frequencies ω correspond to small times t . The asymptotic values for large ω are

$$(\eta^*, V^2, \alpha^2) = \left(\mu, \frac{2\omega\mu}{\rho}, \frac{\omega\rho}{2\mu} \right), \quad (2.9)$$

corresponding to a Newtonian fluid. If, on the other hand, $\mu/\tilde{\mu} = \epsilon$ is small, ω large and

$$1 \ll \lambda^2 \omega^2 \ll \frac{1}{\epsilon^2},$$

then

$$V^2 = c^2 + O(\epsilon), \quad c^2 = \frac{\eta}{\lambda\rho}, \quad (2.10)$$

is the limiting wave speed for a Maxwell model. If, in addition $\lambda^2 \omega^2 \ll 1/\epsilon$, Then the attenuation $-(\alpha^2)^{\frac{1}{2}}$ for large ω is also that for a Maxwell model. In this case the effective modulus is η/λ and it can be obtained by measurements at large frequencies, which are sufficiently small so as not to excite the fast (here, viscous) modes (see §3).

The existence of slow and fast modes, of different times of relaxation, corresponding to different mechanisms of relaxation, is suppressed by the notion of a mean relaxation time. The equation $\eta = G(0)\bar{\lambda}$ cannot be a very useful description of stress relaxation when there are very different times of relaxation. The relaxation from the huge glassy modulus $G(0)$ (say 10^9 Pa) is very fast and the area under this fast mode gives rise only to a very small fraction of the total viscosity. Typically the total viscosity is associated with the slow relaxation $\lambda_\mu \gg \bar{\lambda}$ of modes associated with a much smaller modulus $G_\mu(0)$ (see figure 1).

3. Propagation of shear waves into a semi-infinite region

The theory of the wave-speed meter requires that one first understands some ideas based on the analysis of propagation of waves of shear into viscoelastic fluids at rest.

3.1. Stokes' first problem for viscoelastic liquids

A planar shear flow is a unidirectional velocity field $u = e_x u(y, t)$ which depends only on the coordinate y perpendicular to the direction of motion. Rheological properties of liquids can have a big effect on the way that shear discontinuities propagate. In Stokes' first problem we prescribe a shock in the velocity for boundary data at $y = 0$. We then determine how this data will propagate into the fluid above ($y > 0$) the plate. In addition,

$$u(y, t) = 0 \quad \text{for } t \leq 0 \tag{3.1}$$

is prescribed for all $y \geq 0$. The stress and velocity in $y \geq 0$ satisfy

$$\left. \begin{aligned} \rho \frac{\partial u}{\partial t} &= \frac{\partial \tau}{\partial y}, \\ \frac{\partial \tau}{\partial t} &= \mu \frac{\partial^2 u}{\partial y \partial t} + G(0) \frac{\partial u}{\partial y} + \int_0^t G'(s) \frac{\partial u}{\partial y}(y, t-s) ds, \end{aligned} \right\} \tag{3.2}$$

where

$$u(y, t) \rightarrow 0 \quad \text{as } y \rightarrow \infty.$$

These equations may be combined and written as

$$\rho \frac{\partial^2 u}{\partial t^2} = G(0) \frac{\partial^2 u}{\partial y^2} + \mu \frac{\partial^3 u}{\partial y^2 \partial t} + \int_0^t G'(s) \frac{\partial^2 u(y, t-s)}{\partial y^2} ds. \tag{3.3}$$

At the plate we have

$$u(0, t) = UH(t), \tag{3.4}$$

where $H(t)$ is Heaviside's step function.

When $G(\cdot) = 0$ there is no elasticity and the step input is transmitted to the interior by diffusion with

$$u(y, t) = U \operatorname{erfc} \left[\frac{y}{(4\mu t/\rho)^{1/2}} \right].$$

For elastic waves $\mu = 0$ and (3.3) is like a telegraphers equation. When

$$G(s) = \frac{\eta}{\lambda} e^{-s/\lambda}$$

then (3.3) reduces to

$$\frac{\partial^2 u}{\partial t^2} + \frac{1}{\lambda} \frac{\partial u}{\partial t} = \frac{\eta}{\rho \lambda} \frac{\partial^2 u}{\partial y^2}. \tag{3.5}$$

This telegraphers equation is hyperbolic and transmits waves, but the waves are damped.

It would not be useful to base a rheometrical theory on a special case. It is necessary to understand how shear-wave propagation depends on the relaxation function $G(s)$ and the viscosity μ .

There are many papers on Stokes' first problem, or problems equivalent to it, which are based on Maxwell models. The first group of papers is for elastic fluids $\mu = 0$ with $G(s) = (\eta/\lambda) e^{-s/\lambda}$. The solution of (3.1), (3.4) and (3.5) is given by Carslaw & Jaeger (1947). Step jumps of velocity between two plates have been studied by Böhme (1981), Kazakia & Rivlin (1981), Rivlin (1982, 1983) and Christensen (1982). Stokes'

first problem with $\mu \neq 0$ and $G(s) = (\eta/\lambda) e^{-s/\lambda}$ has been studied by Morrison (1956), Tanner (1962) and Saut & Joseph (1983). Stokes' first problem for fluids with instantaneous elasticity, $\mu = 0$ and a general relaxation function $G(s)$ was studied by Chu (1962), Narain & Joseph (1982, 1983*a, b*) and Renardy (1982) and an allied problem was studied by Coleman, Gurtin & Herrera (1965). All but the authors last mentioned use the method of Laplace transforms and all find that the discontinuity propagates into the fluid at rest with a speed and attenuation given by

$$\left\{ c, u \left(y, \frac{y}{c} \right) \right\} = \left\{ \left(\frac{G(0)}{\rho} \right)^{\frac{1}{2}}, U \exp \left[\frac{yG'(0)}{2cG(0)} \right] \right\}. \tag{3.6}$$

Narain & Joseph (1982) also gave a boundary-layer analysis for small values of μ . The solution of Stokes' first problem with $\mu \neq 0$ and a general relaxation function has recently been obtained and computed by Preziosi & Joseph (1986).

3.2. Scale invariance

We turn next to new results which are important for our analysis of viscosity and rigidity and for the analysis of the wave-speed meter. We shall begin with the general problem when $\mu = 0$ and show how an effective μ arises from the decay of the fast modes. The solution of (3.1)–(3.4) with $\mu = 0$ is given by the Laplace inversion integral (see Narain & Joseph 1982, equations (5.1) and (5.10)):

$$u(y, t; G) = \frac{U}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{d\omega}{\omega} \exp \left[\omega t - y \left(\frac{\rho\omega}{\bar{G}(\omega)} \right)^{\frac{1}{2}} \right], \tag{3.7}$$

where
$$\bar{G}(\omega) = \int_0^\infty G(s) e^{-\omega s} ds \tag{3.8}$$

is the Laplace transform of $G(s)$. We assume that $G(s)$ is positive, smooth and decreasing, and such that $\bar{G}(\omega)$ is analytic in the positive part of the complex- ω plane with the origin excized. All these hypotheses hold, for example, when $G(s)$ is expressed by a discrete relaxation spectrum, as in (2.6), or by an infinite number, or even a continuum of such modes.

The notation used in (3.7) makes explicit that the velocity at point (y, t) is for a fluid with relaxation function $G(s)$. The solution is a functional of $G(\cdot)$.

Our first new result is that the solution of Stokes' first problem satisfies a certain type of group invariance, which we call scaling under radial shifts of y and t . The radial shift is given by

$$(y, t) = (\phi\zeta, \phi\tau), \quad 0 < \phi < \infty. \tag{3.9}$$

The scaling invariance is then described by the following relationship:

$$u(y, t; G) = U(\zeta, \tau; G_\phi), \tag{3.10}$$

where
$$G_\phi(s) = G(\phi s).$$

The scaling invariance says that different observers at different points on a radius in the (y, t) -plane see different scaled relaxation functions.

To prove (3.10) we change variables as in (3.9) and write $\omega = \Omega/\phi$. We find easily that

$$u(y, t; G) = \frac{U}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{d\Omega}{\Omega} \exp \left[\Omega\tau - \zeta \left(\frac{\rho\Omega}{\bar{G}_\phi(\Omega)} \right)^{\frac{1}{2}} \right], \tag{3.11}$$

where
$$\bar{G}_\phi(\Omega) = \int_0^\infty G(\phi s) e^{-\Omega s} ds.$$

3.3. *Asymptotic results*

Various asymptotic results may be derived from (3.10). The strategy is to take (ζ, τ) as $O(1)$ and to derive asymptotic results by taking various limits for ϕ . The first result is for small t and $y, \phi \rightarrow 0$:

$$u(y, t; G) = U \left\{ \exp \frac{yG'(0)}{2cG(0)} + O(\phi y) \right\} H \left(t - \frac{y}{c} \right). \tag{3.12}$$

To prove this we write

$$\begin{aligned} \bar{G}_\phi(\Omega) &= \int_0^\infty e^{-\Omega s} [G(0) + \phi s G'(0) + O(\phi^2 s^2)] ds \\ &= \frac{G(0)}{\Omega} + \frac{\phi G'(0)}{\Omega^2} + O\left(\frac{\phi^2}{\Omega^3}\right). \end{aligned}$$

The integrand for the inversion integral may be written as

$$\begin{aligned} \frac{d\Omega}{\Omega} \exp \left[\Omega \left(\tau - \frac{\zeta}{c} \right) + \frac{\phi \zeta}{c} \frac{G'(0)}{2G(0)} + O\left(\frac{\phi^2}{\Omega}\right) \right] \\ = \frac{d\Omega}{\Omega} \exp \left[\frac{\phi \zeta G'(0)}{2cG(0)} \right] \exp \Omega \left(\tau - \frac{\zeta}{c} \right) \left[1 + O\left(\frac{\phi^2 \zeta}{\Omega}\right) \right]. \end{aligned}$$

After integrating, using the contour appropriate for Heaviside functions (e.g. figure 5.1 in Narain & Joseph 1982), we get (3.12).

We recall that the exact solution of (3.1)–(3.4) with $\mu = 0$ is of the form

$$u(y, t; G) = U f(y, t) H \left(t - \frac{y}{c} \right), \quad f \left(y, \frac{y}{c} \right) = \exp \frac{yG'(0)}{2cG(0)}.$$

The result (3.12) is hardly surprising; when y and t are small we may replace $f(y, t)$ with $f(y, y/c)$.

The second result is for large t and $y, t - y/c > 0$. If $\phi = 1/\epsilon, \epsilon \rightarrow 0$, we find that

$$u(y, t; G) = U \operatorname{erfc} \left[\frac{y}{2 \left(\frac{\eta}{\rho} t \right)^{\frac{1}{2}}} \right] + O(\epsilon). \tag{3.13}$$

To derive this we note that

$$\bar{G}_\phi(\Omega) = \int_0^\infty G \left(\frac{s}{\epsilon} \right) e^{-\Omega s} ds.$$

As $\epsilon \rightarrow 0$ the relaxation function collapses on the origin, like a delta function, and

$$\bar{G}_\phi(\Omega) \rightarrow \int_0^\infty G \left(\frac{s}{\epsilon} \right) ds = \epsilon \eta,$$

provided that $|\Omega|$ is not too large. In fact, the integrand in the inversion integral tends rapidly to zero for large Ω on the contour of integration. Then, with a small error, we get

$$\begin{aligned} u(y, t; G) &= \frac{U}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{d\Omega}{\Omega} \exp \left[\Omega \tau - \zeta \left(\frac{\rho \Omega}{\epsilon \eta} \right)^{\frac{1}{2}} \right], \\ &= \frac{U}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{d\omega}{\omega} \exp \left[\omega t - y \left(\frac{\omega}{\eta/\rho} \right)^{\frac{1}{2}} \right] \\ &= U \operatorname{erfc} \left[\frac{y}{2 \left(\frac{\eta}{\rho} t \right)^{\frac{1}{2}}} \right]. \end{aligned}$$

When t and y are large the solution looks Newtonian with viscosity η . The same type of result holds when $\mu \neq 0$ with η replaced by $\tilde{\mu}$.

3.4. Effective moduli

The third result to come out of scaling invariance shows how an effective viscosity and modulus can arise when different molecular substructures have different times of relaxation. We shall suppose that there is an additive decomposition for $G(s)$, as in (2.7), with

$$G(s) = G_0(s) + G_\mu(s), \tag{3.14}$$

where λ_0 is a short mean time of relaxation and

$$\mu = \int_0^\infty G_0(s) ds. \tag{3.15}$$

We find that when $t \gg \lambda_0, t - y/c > 0$, then with a small error

$$u(y, t; G) = \frac{U}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{d\omega}{\omega} \exp \left[\omega t - \left(\frac{\rho\omega}{\mu + \bar{G}_\mu(\omega)} \right)^{\frac{1}{2}} \right], \tag{3.16}$$

where $\bar{G}_\mu(\omega)$ is the Laplace transform of $G_\mu(s)$. This is the solution of (3.1)–(3.4) when $\mu \neq 0$ and $G(s) = G_\mu(s)$. The elastic response of unrelaxed modes is smoothed by the viscosity μ induced by modes already relaxed.

A heuristic argument for (3.16), which can be made precise, starts from

$$u(\zeta, \tau; G_\phi) = \frac{U}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{d\Omega}{\Omega} \exp \left[\Omega\tau - \zeta \left(\frac{\rho\Omega}{\bar{G}_0^\phi(\Omega) + \bar{G}_\mu^\phi(\Omega)} \right)^{\frac{1}{2}} \right], \tag{3.17}$$

where

$$\bar{G}_0^\phi(\Omega) = \int_0^\infty G_0(\phi s) e^{-\Omega s} ds$$

and

$$\bar{G}_\mu^\phi(\Omega) = \int_0^\infty G_\mu(\phi s) e^{-\Omega s} ds.$$

The function $G_0(s)$ has a mean relaxation time λ_0 and $\phi > \lambda_0$. Hence, $G_0(\phi s)$ is crowded at the origin like a δ -function, nearly zero for very small values of s for which $e^{\Omega s} \sim 1$. Then, with a small error

$$\bar{G}_0^\phi(\Omega) = \frac{\mu}{\phi}$$

and (3.16) follows after rescaling. Small errors are associated with large values of $|\Omega|$ on the line of integration $\text{Re } \Omega = \gamma$. The integrand is very small, and also oscillates rapidly when $|\Omega|$ is large. Of course, $G_\mu(s)$ could still be glassy at times large enough for $G_0(s)$ to have relaxed completely.

3.5. Perturbed elasticity

Having started the study of the problem with $\mu = 0$, we turn now to the study of Stokes' first problem when the viscosity ratio

$$J \stackrel{\text{def}}{=} \frac{\mu}{\tilde{\mu}} \quad (0 \leq J \leq 1)$$

is small. Our aim is to establish the shock-layer approximation

$$u(y, t, \mu; G_\mu) = f(y, t) \int_{-\infty}^\xi e^{-\psi^2} d\xi + o(J), \tag{3.18}$$

where
$$\xi = \frac{t-y/\hat{c}}{(Jyc)^{\frac{1}{2}}}, \tag{3.19}$$

$$\hat{c} = \left(\frac{G_\mu(0)}{\rho[1-J]} \right)^{\frac{1}{2}} = \frac{c_\mu}{(1-J)^{\frac{1}{2}}} \tag{3.20}$$

and

$$u(y, t, 0; G_\mu) = Uf(y, t) H\left(\frac{t-y}{c_\mu}\right),$$

$$f\left(y, \frac{y}{c_\mu}\right) = \exp\left[\frac{yG'_\mu(0)}{2c_\mu G(0)}\right].$$

This result holds for sufficiently small values of $\rho^2 Jyc_\mu/\tilde{\mu}\eta$ and J .

The viscosity ratio arises from the following change to dimensionless variables:

$$[v, \tau, x, g(\tau)] = \left[\frac{u}{U}, \frac{tG_\mu(0)}{\eta}, y\left(\frac{G_\mu(0)\rho}{\eta\mu}\right)^{\frac{1}{2}}, \frac{G_\mu(t)}{G_\mu(0)} \right]. \tag{3.21}$$

Clearly, $g(0) = 1$. After introducing this change of variables into (3.1), (3.3) and (3.4), using the effective modulus we find that $v(x, \tau, J)$ satisfies

$$\left. \begin{aligned} v_{\tau\tau} &= (1-J)v_{xx} + Jv_{x\tau\tau} + (1-J) \int_0^\tau g'(s)v_{xx}(x, \tau-s) ds, \\ v(0, \tau) &= H(\tau), \\ v(x, \tau) &= 0 \quad \text{for } \tau \leq 0. \end{aligned} \right\} \tag{3.22}$$

We note that $1-J = \eta/\tilde{\mu}$. When $J = 0$,

$$\left. \begin{aligned} v(x, \tau, 0) &= f(x, \tau) H(\tau-x), \\ f(x, x) &= e^{\frac{1}{2}xg'(0)}, \\ f(0, \tau) &= 1. \end{aligned} \right\} \tag{3.23}$$

The function $f(x, \tau)$ is defined, up to the change (3.21) to dimensionless variables, by the expression for $f(x, t)$ given by equation 5.10 of Narain & Joseph (1982).

For the Jeffreys' model $g(s) = e^{-s}$ and

$$v_{\tau\tau} + v_\tau - v_{xx} = Jv_{x\tau\tau}.$$

In this case the viscosity ratio is the only parameter in the problem. The viscosity ratio $J = a$, where $a = \lambda_2/\lambda_1$, is the ratio of relaxation upon retardation time, as used by Tanner (1962).

The effect of a small viscosity is to smooth the shock. The amplitude of the shock decays rapidly and eventually the solution looks diffusive, with viscosity $\tilde{\mu}$. We look for smoothing at small values of (x, ϵ) where the unperturbed problem has a large $f(x, \tau)$ and a large shock. We follow Narain & Joseph (1982) and change variables

$$x = JX, \quad t = JT, \quad V(X, T, J) = U(x, t, J),$$

where

$$\left. \begin{aligned} V_{TT} &= V_{XX} + V_{XXT} + JF(X, T, J), \\ V(0, T) &= H(T), \\ V(X, T) &= 0, \quad T \leq 0 \end{aligned} \right\} \tag{3.24}$$

and

$$F(X, T, J) = V_{XX}(X, T, J) + (1-J) \int_0^T g'(Js) V_{XX}(X, T-s) ds.$$

We look at small values of x and t in the limit $J \rightarrow 0$. Assuming that $g'(0)$ is finite, we find that $V(X, T, 0)$ satisfies

$$\left. \begin{aligned} V_{TT} &= V_{XX} + V_{XXT}, \\ V(0, T) &= H(T), \\ V(X, T) &= 0, \quad T \leq 0. \end{aligned} \right\} \quad (3.25)$$

This problem governs the perturbation of Stokes' first problem for pure, rather than relaxing, elasticity. An exact solution of (3.25) was given by Morrison (1956, equation A 14). This solution smooths the Heaviside function. M. Renardy (private communication; to appear in Hrusa, Nohel, Renardy 1986) has studied the

$$\left. \begin{aligned} U_{\tau\tau} &= U_{xx} + \epsilon U_{xx\tau}, \\ U(0, \tau, \epsilon) &= H(t), \\ U(x, \tau, \epsilon) &= 0, \quad \tau \leq 0, \end{aligned} \right\} \quad (3.26)$$

which arises from changing variables $[X, T, V] = [(x/\epsilon), (\tau/\epsilon), U]$. He estimates the Laplace inversion integral for (3.26) to obtain a small- ϵ approximation for (3.26) under the following change of variables:

$$x = x, \quad \xi = \frac{\tau - x}{(\epsilon x)^{\frac{1}{2}}}, \quad (3.27)$$

and he shows that with a small error

$$U(x, \tau, \epsilon) = \int_{-\infty}^{\xi} e^{-\frac{1}{2}\psi^2} d\psi = w(\xi) \quad (3.28)$$

depends on ξ alone, provided that $(\epsilon x)^{\frac{1}{2}}$ is small. Note that

$$w(-\infty) = 0, \quad w(\infty) = 1. \quad (3.29)$$

Clearly

$$\lim_{\epsilon \rightarrow 0} w(\xi) = H(t - x).$$

Renardy's result may also be obtained directly from (3.26) using the change of variables (3.27). We find that when ϵx is small and x -derivatives are relatively small

$$w_{\xi\xi\xi} + \xi w_{\xi\xi} + w_{\xi} = 0, \quad (3.30)$$

subject to (3.29). Equation (3.28) represents the effect of diffusion in smoothing the propagating step function. This analysis shows that the size of the shock layer perturbing the propagating step scales with $(\epsilon x)^{\frac{1}{2}}$.

We may now identify $\epsilon = J$; note that

$$U(x, \tau, J) \approx f(x, \tau) w(\xi), \quad (3.31)$$

with the $f(x, \tau)$ as in (3.23) also satisfies (3.30), the boundary conditions and initial conditions to lowest order in J . We note next that (3.31) is not uniformly valid; it is not exactly correct, for example, at $x = 0$ when $J \neq 0$. A composite shock-layer solution, given by Narain & Joseph (1982, figure 18.1) with αx replaced by $(Jx)^{\frac{1}{2}}$, eliminates this non-uniformity.

4. Mathematical model of the wave-speed meter

We consider two long concentric cylinders of length \hat{L} and radius b and a , $b > a$. Fluid fills the gap $b - a$. At some instant the outer cylinder is forced to rotate with an angular velocity given arbitrarily as $\Omega(t)$, $t \geq 0$. The inner cylinder is free to rotate

on its axis, without friction. Eventually, the shear induced by the outer cylinder causes the inner cylinder to move. We want to determine how the data propagate from the outer cylinder to the inner one, the conditions under which the propagation is dominated by shear waves and the dependence of the speed of propagation on material parameters. The theory that we shall present is meant to guide rheometrical devices using shear waves into regions at rest in the same sense that the theory of small-amplitude sinusoidal oscillations guides devices used to measure the complex viscosity.

The linearized mathematical problem for the device just described, neglecting end effects, can be expressed relative to polar cylindrical coordinates (r, θ, z) as follows:

$$\left. \begin{aligned} \rho \frac{\partial w(r, t)}{\partial t} &= \frac{\partial T^{\langle r\theta \rangle}}{\partial r} + \frac{2}{r} T^{\langle r\theta \rangle}, \\ w(r, t) &= 0, \quad t \leq 0, \end{aligned} \right\} a \leq r \leq b. \quad (4.1)$$

The input at the outer cylinder $r = b$ is prescribed:

$$w(b, t) = \begin{cases} b\Omega(t), & t > 0, \\ 0, & t < 0. \end{cases} \quad (4.2)$$

The output at the inner cylinder is that the fluid and cylinder velocities are the same at $r = a$,

$$a\dot{\theta} = \frac{\partial w(a, t)}{\partial t}, \quad (4.3)$$

and that the shear stress $T^{\langle r\theta \rangle}(a, t)$ on the cylinder gives rise to the torque which turns the cylinder:

$$I\ddot{\theta} = 2\pi a^2 L T^{\langle r\theta \rangle}(a, t), \quad (4.4)$$

where L is the filling level and I is the moment of inertia.

To complete the statement of the problem we need to relate w and $T^{\langle r\theta \rangle}$. We shall for the moment adopt Boltzmann's generalization

$$T^{\langle r\theta \rangle}(r, t) = \int_0^\infty G(s) \left[\frac{\partial w}{\partial r} - \frac{w}{r} \right](r, t-s) ds, \quad (4.5)$$

of Maxwell's equation. Our understanding, as in §2, is that the spectrum of $G(s)$ may include glassy modes with high rigidities that in the absence of slow modes, like those for polymer solutions, simulate Newtonian fluids. We may combine (4.3), (4.4) and (4.5) to give

$$I \frac{\partial w}{\partial t}(a, t) = 2\pi a^3 L \int_0^t G(s) \left[\frac{\partial w}{\partial r} - \frac{w}{r} \right](a, t-s) ds. \quad (4.6)$$

The linearized dynamics of the wave speed is governed by (4.1), (4.2) and (4.6). The same problem but with (4.6) replaced by $w(a, t) = 0$ was solved by Narain & Joseph (1982). Here we shall reduce this new problem to the old one and derive some previously unspecified properties of the old problem.

Since the gap $d/b \ll 1$ in the wave-speed meter and it is these small gaps which are of greatest interest, we may seek a simplified problem for small d/b .

5. The narrow-gap approximation for short times

This approximation is associated with the following physics. A shear wave is initiated at the outer cylinder. When this wave hits the inner cylinder, the velocity at $r = a$ is initially zero, the shear velocity that hits $r = a$ is annihilated by the reflected wave at the time of first reflection, but the shear stress at $r = a$ doubles and this provides the torque that turns the inner cylinder. This leads to a problem that satisfies $w(a, t) = 0$ at the short time of first reflection. This problem defines a family of initial-value problems for an elastic fluid which includes Stokes' first problem. The equation of motion (4.6) for the cylinder is solved as an afterthought, with the first approximation to w on the left and the zeroth order on the right.

We shall now derive the physical result described in the last paragraph. We define

$$v(y, t) = w(r - t), \quad y = b - r, \tag{5.1}$$

and seek $v(y, t)$ as a series in $\delta = d/b$, valid for short times in the neighbourhood of the first time of reflection:

$$v(y, t) = v_0(y, t) + \delta v_1(y, t) + \delta^2 v_2(y, t) + \dots \tag{5.2}$$

To derive the problems satisfied by v_0, v_1 , etc. we introduce the fast dimensionless times τ and σ

$$t = \frac{\delta \tau}{\Omega_0}, \quad s = \frac{\delta \sigma}{\Omega_0} \tag{5.3}$$

and

$$\left. \begin{aligned} y &= dx, \\ v(y, t) &= b\Omega_0 u(x, \tau) \end{aligned} \right\} \tag{5.4}$$

where Ω_0 is the largest value of $\Omega(s)$. We first determine that

$$\begin{aligned} \rho(b\Omega_0)^2 \frac{\partial u}{\partial \tau} &= \int_0^\tau \hat{G}(\sigma) \frac{\partial^2 u}{\partial x^2}(x, \tau - \sigma) d\sigma - \frac{\delta}{1 - \delta x} \int_0^\tau \hat{G}(\sigma) \frac{\partial u}{\partial x}(x, \tau - \sigma) d\sigma \\ &\quad - \left[\frac{\delta}{1 - \delta x} \right]^2 \int_0^\tau \hat{G}(\sigma) u(x, \tau - \sigma) d\sigma, \quad 0 \leq x \leq 1, \end{aligned} \tag{5.5a}$$

$$\begin{aligned} u(x, \tau) &= 0, \quad \tau \leq 0, \\ u(0, \tau) &= \begin{cases} \hat{\Omega}(\tau), & \tau > 0 \\ 0, & \tau < 0, \end{cases} \end{aligned} \tag{5.5b}$$

$$\frac{\partial u(1, \tau)}{\partial \tau} = -\frac{2\pi a^3 L}{bI\Omega_0^2} \left[\delta \int_0^\tau \hat{G}(\sigma) \frac{\partial u}{\partial x}(1, \tau - \sigma) d\sigma + \frac{\delta^2}{1 - \delta} \int_0^\tau \hat{G}(\sigma) u(1, \tau - \sigma) d\sigma \right], \tag{5.5c}$$

where $\hat{\Omega}(\tau) = \Omega(t)/\Omega_0$ and

$$\hat{G}(\sigma) \stackrel{\text{def}}{=} G(s) = G\left[\frac{\delta \sigma}{\Omega_0}\right]. \tag{5.6}$$

Short times are times $\Delta t < \delta/\Omega_0$. The main effect of our scaling is to introduce δ as a factor on the right of (5.5c). We now expand

$$u(x, \tau) = u_0 + \delta u_1 + \delta^2 u_2 + \dots \tag{5.7}$$

and identify the coefficients of powers of δ . At zeroth order

$$\left. \begin{aligned} \rho(b\Omega_0)^2 \frac{\partial u_0}{\partial \tau} &= \int_0^\tau \hat{G}(\sigma) \frac{\partial^2 u_0}{\partial x^2}(x, \tau - \sigma) d\sigma, \quad 0 \leq x \leq 1 \\ u_0(x, \tau) &= 0, \quad \tau \leq 0, \\ u_0(0, \tau) &= \begin{cases} \tilde{Q}(\tau), & \tau > 0 \\ 0, & \tau < 0, \end{cases} \\ u_0(1, \tau) &= 0, \quad \tau \geq 0. \end{aligned} \right\} \quad (5.8)$$

This is a generalization of Stokes' first problem in which arbitrary initial data replace a unit step jump. At first order, we find that

$$\rho(b\Omega_0)^2 \frac{\partial u_1}{\partial t} = \int_0^\tau \hat{G}(\sigma) \left[\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial u_0}{\partial x} \right](x, \tau - \sigma) d\sigma \quad (5.9a)$$

and
$$u_1(x, \tau) = 0, \tau \leq 0 \quad \text{in} \quad 0 \leq x \leq 1, \quad (5.9b)$$

while
$$u_1(0, \tau) = 0 \quad \text{for all } \tau \geq 0. \quad (5.9c)$$

The motion of the inner cylinder is then governed by

$$\frac{\partial u_1}{\partial \tau}(1, \tau) = -\frac{2\pi a^3 L}{Ib\Omega_0^2} \int_0^\tau \hat{G}(\sigma) \frac{\partial u_0}{\partial x}(1, \tau - \sigma) d\sigma. \quad (5.10)$$

Equations (5.9) and (5.10) determine $u_1(x, \tau)$. $u_1(1, \tau)$ may be found from (5.10).

We have found that for small times the motion of the inner cylinder is determined by (5.10) where $u_0(x, \tau)$ satisfies (5.8). This perturbation can be established rigorously using Laplace transforms. The perturbation is not valid for long times.

6. Response of the inner cylinder to a step change of velocity of the outer cylinder

Ideally we should like to be able to move the outer cylinder impulsively, with

$$\tilde{Q}(\tau) = 1, \quad \tau > 0 \quad (6.1)$$

in (5.5b). The problem (5.8) with unit-step-function data (6.1) was solved by Narain & Joseph (1982). They found (cf. their (8.7)) that

$$\begin{aligned} u_0(x, \tau) &= f(x, \tau) H(\tau - \hat{\alpha}x) + \{f(x + 2, \tau) H[\tau - \hat{\alpha}(x + 2)] \\ &\quad - f(2 - x, \tau) H[\tau - \hat{\alpha}(2 - x)]\} + \dots, \end{aligned} \quad (6.2)$$

where
$$\frac{1}{\hat{\alpha}} \stackrel{\text{def}}{=} \hat{c} = \frac{1}{b\Omega_0} \left(\frac{\hat{G}(0)}{\rho} \right)^{\frac{1}{2}}$$

is a dimensionless wave speed and

$$f(x, \hat{\alpha}x^+) = \exp \left[\frac{\hat{\alpha}x \hat{G}'(0)}{2\hat{G}(0)} \right]. \quad (6.3)$$

The solution (6.2) actually describes the development of steady Couette flow through multiple reflections. It is a valid solution of our problem only at early times near to the time $\tau = \hat{\alpha}$ of first reflection, that is when

$$t = \frac{b-a}{c}, \quad c = \left(\frac{G(0)}{\rho} \right)^{\frac{1}{2}}.$$

The velocity $u_0(1, \hat{\alpha}) = 0$, but the stress doubles at $x = 1$ and $\tau = \hat{\alpha}$ (cf. equation (3.10) of Narain & Joseph 1983*a*).

To find the motion of the inner cylinder (at $x = 1$) in the interval $(\hat{\alpha}, 2\hat{\alpha})$ of time τ between the first and second reflection, we evaluate (5.10), using (6.2):

$$\frac{\partial u_0}{\partial x}(1, \tau) = -2\hat{\alpha}f(1, \tau)\delta(\tau - \hat{\alpha}) + 2\frac{\partial f}{\partial x}(1, \tau)H(\tau - \hat{\alpha}),$$

where $\delta(\tau - \alpha)$ is Dirac's δ -function. Inserting this into (5.10) we find

$$\frac{\partial u_1(1, \tau)}{\partial \tau} = \frac{4\pi a^3 L}{Ib\Omega_0^2} \left[\hat{\alpha}G(\tau - \hat{\alpha})f(1, \hat{\alpha}^+) - \int_0^{\tau - \hat{\alpha}} G(\sigma)\frac{\partial f}{\partial x}(1, \tau - \sigma) d\sigma \right]. \quad (6.4)$$

Equation (6.4) is the differential equation which governs the motion of the inner cylinder. The wave-speed meter requires that we know $u_1(1, \tau)$ for very small values of $\tau - \hat{\alpha} \geq 0$. In this time interval we may write (6.4) as

$$\frac{\partial u_1(1, \tau)}{\partial \tau} = A_0 + A_1(\tau - \hat{\alpha}) + O(|\tau - \hat{\alpha}|^2), \quad (6.5)$$

where $A_0 = \frac{4\pi a^3 L}{Ib\Omega_0^2} \hat{\alpha}G(0) \exp\left[\frac{\hat{\alpha}G'(0)}{2G(0)}\right],$

$$A_1 = \frac{4\pi a^3 L}{Ib\Omega_0^2} \left[\hat{\alpha}G'(0) \exp\left[\frac{\hat{\alpha}G'(0)}{2G(0)}\right] - G(0)\frac{\partial f}{\partial x}(1, \hat{\alpha}^+) \right],$$

where $(\partial f/\partial x)(1, \hat{\alpha}^+)$ is given in known terms by equation (1.5) of Narain & Joseph (1983*b*).

Returning now to physical variables we may write

$$\frac{\partial w(a, t)}{\partial t} = \frac{b^2}{d} \Omega_0^2 \frac{\partial u(1, \tau)}{\partial \tau}.$$

Hence, using (5.6) and (5.7),

$$\frac{\partial w(a, t)}{\partial(t - d/c)} = \frac{4\pi a^3 L b \Omega_0 (\rho G(0))^{\frac{1}{2}}}{I} \exp\left[\frac{dG'(0)}{2cG(0)}\right] + O\left[\left|t - \frac{d}{c}\right|\right] + O\left[\left(\frac{d}{b}\right)^2\right]. \quad (6.6)$$

Since $w(a, t) = a\theta$, we have

$$\theta(t) = \frac{4\pi^2 L b \Omega_0 (\rho G(0))^{\frac{1}{2}}}{I} \exp\left[\frac{dG'(0)}{2cG(0)}\right] \frac{1}{2} \left[t - \frac{d}{c} \right]^2 + O\left[\left|t - \frac{d}{c}\right|^3\right] + O\left[\left(\frac{d}{b}\left(t - \frac{d}{c}\right)\right)^2\right]. \quad (6.7)$$

This useful formula shows how the inner cylinder moves after being hit by a wave.

It is of interest to compare the response of the inner cylinder to step data of the form (6.1) for an elastic fluid and a Newtonian fluid. Solutions for Stokes' problem in a semi-infinite region and between rigid walls are well known. We write a formula for the torque on the inner cylinder at $r = a$ using these two solutions and integrate the equation of motion for the inner cylinder. At small times this gives

$$\theta(t) = \frac{8a^2 L b \Omega_0 \rho (\pi \nu)^{\frac{1}{2}}}{3I} t^{\frac{3}{2}} + o(t^{\frac{3}{2}})$$

for the problem between parallel plates and one-half of this value for the problem in a semi-infinite region. The displacement starts at the instant that the outer cylinder is moved. There is no transit time, no delay, under the assumptions of this analysis.

7. Response of the inner cylinder to a smooth change of velocity of the outer cylinder

The kicking mechanism that moves the outer cylinder of the wave-speed meter does not lead to a step jump of velocity as was assumed in §6. Instead there is a smooth rise which is recorded from oscilloscope traces shown in §1 of Part 2. We have solved the problem for arbitrary smooth inputs. We modelled the actual input and showed that the response for this model differs only slightly from the response for the step jump when $d/b \ll 1$ and $\beta/\lambda_\mu \ll 1$, where β is the time the outer cylinder needs to accelerate to a constant angular velocity and λ_μ is an effective time of relaxation for an effective modulus (see figure 1), which we modelled with a single exponential $G(s) = G_\mu e^{-s/\lambda_\mu}$.

8. Nomenclature

This section contains definitions of symbols used in this paper, omitting those that are locally defined, and some common symbols with clear meanings.

c	Shear-wave speed, speed of a discontinuity propagating into a region at rest
$d = b - a$	Gap size
$\mathbf{D}[\mathbf{u}]$	Symmetric part of the velocity gradient
$G(s), G_\mu(s)$	Shear relaxation functions
$G(0), G_\mu(0)$	Shear moduli

$G_\mu(0) \stackrel{\text{def}}{=} G_c$ Effective shear modulus, $G_c = \rho c^2$ is experimentally determined

It should be noted that rheologists usually call the relaxation function a relaxation modulus. We use the word modulus for a value of this function.

$\mathbf{G}(s) = \mathbf{C}_t(s) - \mathbf{1}$	The right relative Cauchy–Green tensor minus the identity tensor
$\tilde{G}'(\omega), \tilde{G}''(\omega)$	Storage modulus, loss modulus. These quantities and the complex viscosity are obtained in dynamic measurements using small-amplitude oscillations
$J = \mu/\tilde{\mu}$	Viscosity ratio
$\alpha = \frac{1}{c}, \hat{\alpha} = \frac{1}{c} b\Omega_0$	
$\eta = \int_0^\infty G(s) ds$	Elastic viscosity
η^*	Complex viscosity
μ	Newtonian viscosity
$\tilde{\mu} = \mu + \eta$	Zero shear or static viscosity
ω	Radian frequency
$\Omega(t)$	Angular velocity of the outer cylinder
Ω_0	Maximum value of $\Omega(t)$

Important times:

$\Delta t = \alpha d$	Transit time†
$\Delta t_o, \Delta t_c$	Transit time, measured on the oscilloscope and on the electronic counter, respectively†
$\Delta t_a, \Delta t_b$	Fall times of the voltage at the photodiodes corresponding to the inner and outer cylinder respectively†
β	Rise time of the input signal
ϵ	Response time of the experiment defined by $G_c = G(\epsilon)$
$\lambda, \lambda_0, \lambda_\mu$	Relaxation times (λ_0 for glassy modes, λ_μ for viscoelastic modes)

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† Indicates quantities used in Part 2.

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